

# UNIFORM DISCONNECTEDNESS AND QUASI-ASSOUAD DIMENSION

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**ABSTRACT.** The uniform disconnectedness is an important invariant property under bi-Lipschitz mapping, and the Assouad dimension  $\dim_A X < 1$  implies the uniform disconnectedness of  $X$ . According to quasi-Lipschitz mapping, we introduce the quasi-Assouad dimension  $\dim_{qA}$  such that  $\dim_{qA} X < 1$  implies its quasi uniform disconnectedness. We obtain  $\overline{\dim}_B X \leq \dim_{qA} X \leq \dim_A X$  and compute the quasi-Assouad dimension of Moran set.

## 1. INTRODUCTION

A subset  $E$  of metric space  $(X, d)$  is said to be **uniformly disconnected** [4], if there is a constant  $0 < c < 1$  such that for any  $x \in E$  and any  $0 < r < r^*$  for some  $r^*$ , there exists a set  $E_{x,r} \subset E$  satisfying

$$E \cap B(x, cr) \subset E_{x,r} \subset B(x, r) \text{ and } \text{dist}(E_{x,r}, E \setminus E_{x,r}) \geq cr, \quad (1.1)$$

where  $B(x, r)$  is the closed ball with center  $x$  and radius  $r$ , and  $\text{dist}(\cdot, \cdot)$  denotes the least distance between sets. This uniform disconnectedness is an invariant property under any bi-Lipschitz mapping. Here  $f : (X, d_X) \rightarrow (Y, d_Y)$  is bi-Lipschitz, if there exists a constant  $L > 0$  such that for all  $x_1, x_2 \in X$ ,

$$L^{-1}d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq Ld_X(x_1, x_2). \quad (1.2)$$

The uniform disconnectedness plays an important role in fractal geometry. David and Semmes [4] obtained the uniformization theorem on quasisymmetric equivalence: if a compact metric spaces is uniformly disconnected, uniformly perfect and doubling, then they are quasisymmetrically equivalent to the Cantor ternary set  $C$ . Mattila and Saaranen [20] proved that suppose  $E$  and  $F$  are Ahlfors-David  $s$ -regular and  $t$ -regular respectively with  $s < t$ , if  $E$  is uniformly disconnected, then  $E$  can be bi-Lipschitz embedded into  $F$ . Based on the method of [20], Wang and Xi [26] discussed the quasi-Lipschitz equivalence between uniformly disconnected Ahlfors-David regular sets.

An interesting property (e.g. see Proposition 5.1.7 of [18]) is that

$$\dim_A X < 1 \implies X \text{ is uniformly disconnected.} \quad (1.3)$$

We can recall **Assouad dimension**  $\dim_A$  as follows. We say  $(X, d)$  is *doubling* if there exists an integer  $N > 0$  such that each closed ball in  $X$  can be covered by  $N$  closed balls of half the radius. Repeated applying the doubling property, it

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gives that there exist constants  $b, c > 0$  and  $\alpha > 0$  such that for all  $r$  and  $R$  with  $0 < r < R < b$ , every closed ball  $B(x, R)$  can be covered by  $c(\frac{R}{r})^\alpha$  balls of radius  $r$ . Let  $N_{r,R}(X)$  denote the smallest number of balls with radii  $r$  needed to cover any ball with radius  $R$ . The Assouad dimension of  $X$ , denoted by  $\dim_A X$ , is defined as

$$\dim_A X = \inf\{\alpha \geq 0 \mid \exists b, c > 0 \text{ s.t. } N_{r,R}(X) \leq c(\frac{R}{r})^\alpha \forall 0 < r < R < b\},$$

which was introduced by Assouad in the late 1970s [1, 2, 3]. Now it plays a prominent role in the study of **quasiconformal mappings** and **embeddability problems**, and we refer the readers to the textbook [8] and the survey paper [14] for more details. Olsen [23] obtained the Assouad dimensions for a class of fractals with some flexible graph-directed construction, Mackay [17] and Fraser [7] calculated the Assouad dimensions of some classes of self-affine fractals. It is well known that  $\dim_H X \leq \bar{\dim}_B X \leq \dim_A X$ , where  $\dim_H(\cdot)$  and  $\bar{\dim}_B(\cdot)$  are Hausdorff and upper box dimensions respectively. For example, if  $E$  is Ahlfors-David  $s$ -regular [4], then  $\dim_A E = \dim_H E = s$ , furthermore, if  $s < 1$  then  $E$  is uniformly disconnected.

We say that a bijection  $f : X \rightarrow Y$  is a **quasi-Lipschitz** mapping, if for all  $x_1, x_2 \in X$ ,

$$\frac{\log d_Y(f(x_1), f(x_2))}{\log d_X(x_1, x_2)} \rightarrow 1 \text{ uniformly as } d_X(x_1, x_2) \rightarrow 0. \quad (1.4)$$

We say  $X$  and  $Y$  are quasi-Lipschitz equivalent, if the above quasi-Lipschitz mapping exists. We can introduce quasi-Hölder equivalence and mapping, if 1 in (1.4) is replaced by  $\dim_H X / \dim_H Y$ .

Inspired by [20], Wang and Xi [27] introduced the quasi Ahlfors-David regularity and **quasi uniform disconnectedness** and proved that  $E$  is quasi-Hölder equivalent to the Cantor ternary set  $C$  if and only if  $E$  is quasi Ahlfors-David regular and quasi uniformly disconnected. As a consequence, they obtained that if quasi Ahlfors-David regular sets  $E$  and  $F$  are quasi uniformly disconnected, then  $E$  and  $F$  are quasi-Lipschitz equivalent if and only if  $\dim_H E = \dim_H F$ .

**Definition 1.** We say that a compact subset  $E$  of metric space  $X$  is quasi uniformly disconnected, if there is a function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\psi(r) < r$  for  $r > 0$  and  $\lim_{r \rightarrow 0} \frac{\log \psi(r)}{\log r} = 1$  such that for any  $x \in E$  and any  $0 < r < r^*$  ( $r^*$  is a constant), there exists a set  $E_{x,r} \subset E$  satisfying

$$E \cap B(x, \psi(r)) \subset E_{x,r} \subset B(x, r) \text{ and } \text{dist}(E_{x,r}, E \setminus E_{x,r}) \geq \psi(r). \quad (1.5)$$

Note that the quasi uniform disconnectedness is an invariant property under any quasi-Hölder(Lipschitz) mapping.

**Example 1.** Given  $0 < \alpha < 1$ . Let  $a_k = \prod_{i=1}^k (1 - (i+1)^{-\alpha})$  for all  $k \geq 1$ . We can check that the countable set  $E = \{0, 1, a_1, a_2, \dots\}$  is quasi uniformly disconnected but not uniformly disconnected. We give the detail of this example in Section 2.

### 1.1. Quasi-Assouad dimension.

The motivation of this manuscript is to introduce a notion named quasi-Assouad dimension  $\dim_{qA} X$  satisfying that

$$\dim_{qA} X < 1 \implies X \text{ is quasi uniformly disconnected.}$$

We will also compute the quasi-Assouad dimension for Moran set.

**Definition 2.** For any  $\delta \in (0, 1)$ , let

$$h_X(\delta) = \inf\{\alpha \geq 0 : \exists b, c > 0 \text{ s.t. } N_{r,R}(X) \leq c \left(\frac{R}{r}\right)^\alpha \quad \forall 0 < r < r^{1-\delta} \leq R < b\}.$$

Then the quasi-Assouad dimension  $\dim_{qA} X$  is defined by

$$\dim_{qA} X = \lim_{\delta \rightarrow 0} h_X(\delta).$$

It is easy to check that

- (a)  $\dim_{qA} E \leq \dim_{qA} F$  if  $E \subset F$ ;
- (b)  $\dim_{qA}(E \cup F) = \max(\dim_{qA} E, \dim_{qA} F)$ ;
- (c)  $\dim_{qA} E = \dim_{qA} f(E)$  if  $f$  is a bi-Lipschitz mapping.

Then we have the following proposition.

**Proposition 1.** Suppose quasi-Assouad dimension is defined as above. Then

- (1)  $\overline{\dim}_B X \leq \dim_{qA} X \leq \dim_A X$ ;
- (2)  $\dim_{qA} E = \dim_{qA} g(E)$  if  $g$  is a quasi-Lipschitz mapping;
- (3)  $\dim_{qA} X = \lim_{\delta \rightarrow 0} \overline{\lim}_{r \rightarrow 0} \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}(X)}{\log R - \log r}$ ;
- (4) If  $\dim_{qA} X < 1$ , then  $X$  is quasi uniformly disconnected.

**Remark 1.** There exists a Moran set  $E$  satisfying  $\overline{\dim}_B E < \dim_{qA} E < \dim_A E$ , see Example 3 in the next subsection.

## 1.2. Moran set.

Some special cases of Moran sets were first studied by Moran [22]. The later works [9, 10, 28, 29] developed the theory on the geometrical structures and dimensions of Moran sets systematically.

Suppose that  $J$  is an initial closed interval of  $\mathbb{R}$ . Let  $\{n_k\}_{k \geq 1}$  be an integer sequence satisfying  $n_k \geq 2$  for all  $k$ . Suppose  $c_k \in (0, 1/n_k]$  for all  $k$ . Denote  $\mathcal{D}^k = \{i_1 \cdots i_k : i_t \in \mathbb{N} \cap [1, n_t] \text{ for all } t\}$  and  $\mathcal{D}^0 = \{\emptyset\}$  with empty word  $\emptyset$ . Let  $J_\emptyset = J$ . Suppose for any  $k \geq 1$  and any  $i_1 \cdots i_{k-1} \in \mathcal{D}^{k-1}$ ,  $J_{i_1 \cdots i_{k-1}1}, \dots, J_{i_1 \cdots i_{k-1}n_k}$  are closed subintervals of  $J_{i_1 \cdots i_{k-1}}$  with their interiors pairwise disjoint, such that the ratio

$$\frac{|J_{i_1 \cdots i_{k-1}j}|}{|J_{i_1 \cdots i_{k-1}}|} = c_k \text{ for all } 1 \leq j \leq n_k. \quad (1.6)$$

Then we call the following compact set

$$F = \bigcap_{k=0}^{\infty} \bigcup_{i_1 \cdots i_k \in \mathcal{D}^k} J_{i_1 \cdots i_k} \quad (1.7)$$

a Moran set with the structure  $(J, \{n_k\}_k, \{c_k\}_k)$ . We denote  $F \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$ . By [29], if  $\sup_k n_k < \infty$ , then

$$\dim_H F = \underline{\lim}_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k}, \quad \overline{\dim}_B F = \overline{\lim}_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k} \quad (1.8)$$

**Remark 2.** The Moran structure is quite different from the self-similar structure. In Moran structure, the relative positions of subintervals  $\{J_{i_1 \cdots i_{k-1}j}\}_{j=1}^{n_k}$  in  $J_{i_1 \cdots i_{k-1}}$  can be variant,  $J_{i_1 \cdots i_{k-1}i_k}$  and  $J_{i_1 \cdots i_{k-1}i'_k}$  may share a common endpoint.

If for any  $k$  and any  $i_1 \cdots i_{k-1}$ ,  $J_{i_1 \cdots i_{k-1}1}, \dots, J_{i_1 \cdots i_{k-1}n_k}$  are distributed uniformly in  $J_{i_1 \cdots i_{k-1}}$  such that  $J_{i_1 \cdots i_{k-1}1}$  ( $J_{i_1 \cdots i_{k-1}n_k}$ ) and  $J_{i_1 \cdots i_{k-1}}$  share the left (right) endpoint, we call the Moran set a uniform Cantor set.

For Moran set  $F \in \mathcal{M}(J, \{n_k\}, \{c_k\})$  with  $\inf_k c_k > 0$ , Li, Li, Miao and Xi [13] obtained the Assouad dimension

$$\dim_A F = \lim_{m \rightarrow \infty} \sup_k \frac{\log(n_{k+1} \cdots n_{k+m})}{-\log(c_{k+1} \cdots c_{k+m})}. \quad (1.9)$$

For uniform Cantor set  $K$  with parameters  $(J, \{n_k\}_k, \{c_k\}_k)$ , one recent result by Peng, Wang and Wen [24] is that  $\dim_A K = 1$  if  $\sup_k n_k = +\infty$ .

**Theorem 1.** *If  $\lim_{k \rightarrow \infty} \frac{\log c_k}{\log c_1 \cdots c_k} = 0$ , then for any  $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$ ,*

$$\dim_{qA} E = \lim_{\delta \rightarrow 0} \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} \frac{\log(n_p \cdots n_q)}{-\log(c_p \cdots c_q)},$$

where  $l_{q,\delta} = \max\{1 \leq p \leq q : \frac{\log(c_p \cdots c_q)}{\log(c_1 \cdots c_q)} > \delta\}$ . In particular, if  $\inf_{k \geq 1} c_k > 0$ , we obtain that

$$\dim_{qA} E = \lim_{\eta \rightarrow 0} \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq q(1-\eta)} \frac{\log(n_p \cdots n_q)}{-\log(c_p \cdots c_q)}.$$

**Corollary 1.** *If  $\lim_{k \rightarrow \infty} \frac{\log c_k}{\log c_1 \cdots c_k} = 0$  and  $\overline{\lim}_{k \rightarrow \infty} \frac{\log n_k}{-\log c_k} < 1$ , then any  $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$  is quasi uniformly disconnected.*

The following Moran sets in Examples 2-4 are quasi uniformly disconnected but not uniformly disconnected. In fact, for compact set  $E \subset \mathbb{R}^1$ ,  $\dim_A E < 1$  if and only if  $E$  is uniformly disconnected [14].

**Example 2.** *Consider a uniform Cantor set  $K$  with  $J = [0, 1]$ ,  $n_k = 3^k$ ,  $c_k = 3^{-2k}$  for all  $k$ . Then  $\lim_{k \rightarrow \infty} \frac{\log c_k}{\log c_1 \cdots c_k} = 0$  and  $\sup_k n_k = +\infty$ , using Theorem 1 and the result on Assouad dimension by Peng et al, we have*

$$\frac{1}{2} = \dim_{qA} K < \dim_A K = 1.$$

**Example 3.** *Let  $\{q_1 < q_2 < \cdots < q_t < q_{t+1} < \cdots\}$  be a positive integer sequence such that  $q_{t+1} > 2q_t$  for all  $t$  and  $\lim_{t \rightarrow \infty} \frac{t}{q_t} = \lim_{t \rightarrow \infty} \frac{q_1 + q_2 + \cdots + q_{t-1}}{q_t} = 0$ . Let  $n_k \equiv 2$  and*

$$c_k = \begin{cases} 1/4 & \text{if } k \in (q_t, 2q_t], \\ (1 - \frac{1}{2t})/2 & \text{if } k \in (2q_t, 2q_t + t], \\ 1/5 & \text{otherwise.} \end{cases}$$

According to (1.8), Theorem 1 and (1.9), for any  $E \in \mathcal{M}([0, 1], \{n_k\}, \{c_k\})$  we have

$$\overline{\dim}_B E = \frac{2 \log 2}{\log 5 + \log 4} < \dim_{qA} E = \frac{1}{2} < \dim_A E = 1.$$

**Example 4.** *Let  $\{q_t\}_t$  and  $\{n_k\}_k$  be defined as in Example 3. Suppose  $f : [1, 2] \rightarrow (2, 5)$  is a continuous function. Let*

$$\bar{c}_k = \begin{cases} 1/f(\frac{k}{q_t}) & \text{if } k \in (q_t, 2q_t], \\ (1 - \frac{1}{2t})/2 & \text{if } k \in (2q_t, 2q_t + t], \\ 1/5 & \text{otherwise.} \end{cases}$$

Then for any  $F \in \mathcal{M}([0, 1], \{n_k\}, \{\bar{c}_k\})$  we have

$$\overline{\dim}_B F = \frac{\log 2}{(\log 5 + \int_1^2 \log f(x) dx)/2} < \dim_{qA} F = \frac{\log 2}{\log \left( \min_{1 \leq x \leq 2} f(x) \right)} < \dim_A F = 1.$$

### 1.3. Quasi-Lipschitz equivalence of Moran sets.

We say  $E \in \mathcal{M}(J, \{n_k\}_k, \{c_k\}_k)$  is of slow change, if

$$\lim_{k \rightarrow \infty} \frac{\log c_k}{\log c_1 \cdots c_k} = 0 \text{ and } \inf_k \frac{\log n_k}{\log c_k} > 0. \quad (1.10)$$

In fact, if  $\inf_k c_k > 0$ , then  $\inf_k \frac{\log n_k}{\log c_k} > 0$ . The scale function  $g_E(r)$  of  $E$  is defined by

$$g_E(r) = \frac{\log n_1 \cdots n_k}{-\log c_1 \cdots c_k} \text{ if } c_1 \cdots c_k \leq \frac{r}{|J|} < c_1 \cdots c_{k-1}. \quad (1.11)$$

It is proved in [16] that two quasi uniformly disconnected Moran sets  $E, F$  of slow change are quasi-Lipschitz equivalent if and only if

$$\lim_{r \rightarrow 0} \frac{g_E(r)}{g_F(r)} = 1. \quad (1.12)$$

**Example 5.** Let  $J = [0, 1]$ ,  $n_k \equiv 2$ ,  $c_k \equiv 1/4$ . Then by (1.12), the uniform Cantor set  $K$  in Example 2 and any  $E \in \mathcal{M}([0, 1], \{n_k\}_k, \{c_k\}_k)$  are quasi-Lipschitz equivalent, although their structures seem to be quite different.

**Example 6.** Suppose  $\{q_t\}_t$  and  $\{n_k\}_k$  are given as in Example 3. When  $k \in (q_t, 2q_t + t]$  for some  $t$ , we define  $c_k = d_k$  as in Example 3. When  $k \in (2q_t + t, q_{t+1}]$  for some  $t$ , we can select  $c_k, d_k$  from  $\{1/5, 1/6\}$ . Then  $E \in \mathcal{M}([0, 1], \{n_k\}_k, \{c_k\}_k)$ ,  $F \in \mathcal{M}([0, 1], \{n_k\}_k, \{d_k\}_k)$  are Moran sets of slow change, and they are quasi-Lipschitz equivalent if and only if

$$\lim_{k \rightarrow \infty} \frac{\#\{i \leq k : c_i = 1/5\}}{\#\{i \leq k : d_i = 1/5\}} = 1 \text{ or } \lim_{k \rightarrow \infty} \frac{\#\{i \leq k : c_i = 1/6\}}{\#\{i \leq k : d_i = 1/6\}} = 1.$$

The paper is organized as follows. We prove the basic properties of quasi-Assouad dimension in Section 2. In particular, we use the idea [20] by Mattila and Saaranen to verify (4) of Proposition 1. In section 3, we compute the quasi-Assouad dimension under the assumption  $\lim_{k \rightarrow \infty} \frac{\log c_k}{\log c_1 \cdots c_k} = 0$ . In the last section, we discuss the quasi-Assouad dimension for general Moran sets.

## 2. BASIC PROPERTIES OF QUASI-ASSOUAD DIMENSION

### 2.1. Detail of Example 1.

Note that  $\sum_{i=1}^{\infty} (i+1)^{-\alpha} = \infty$ , then  $\lim_{k \rightarrow \infty} a_k = 0$ .

It is easy to check that  $\{a_i - a_{i+1}\}_i$  is decreasing. We will use the following estimation

- (i)  $\lim_{k \rightarrow \infty} \frac{\log a_k}{\log a_{k-1}} = 1$ ,
- (ii)  $\lim_{k \rightarrow \infty} \frac{\log k}{\log a_k} = 0$ ,
- (iii)  $\lim_{k \rightarrow \infty} \frac{\log(a_k - a_{k+1})/2}{\log a_k} = \lim_{k \rightarrow \infty} \frac{\log(a_{k-1} - a_k)/2}{\log(a_k - a_{k+1})/2} = 1$ .

For (i),  $\lim_{k \rightarrow \infty} \frac{\log a_k}{\log a_{k-1}} = 1 + \lim_{k \rightarrow \infty} \frac{\log(1 - (k+1)^{-\alpha})}{\log a_k} = 1$ . For (ii), by Stolz theorem and (i), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\log k}{-\log a_k} &= \lim_{k \rightarrow \infty} \frac{\log(k+1)}{-\log a_{k-1}} = \lim_{k \rightarrow \infty} \frac{\log \frac{k+1}{k}}{-\log(a_{k-1}/a_{k-2})} \\ &= \lim_{k \rightarrow \infty} \frac{\log(1 + \frac{1}{k})}{-\log(1 - k^{-\alpha})} = \lim_{k \rightarrow \infty} k^{-1+\alpha} = 0. \end{aligned}$$

Since  $a_k - a_{k+1} = (k+2)^{-\alpha} a_k$ , using Stolz theorem and (ii), we obtain (iii).

(1) Firstly, the set  $E$  is not uniformly disconnected and thus  $\dim_A E = 1$ . Otherwise, suppose that there exist constant  $0 < c < 1$  and  $r^* > 0$  such that for any  $x \in E$  and any  $0 < r < r^*$ , we can find a set  $E_{x,r} \subset E$  satisfying

$$E \cap B(x, cr) \subset E_{x,r} \subset B(x, r) \text{ and } \text{dist}(E_{x,r}, E \setminus E_{x,r}) \geq cr.$$

Notice that the gap sequence  $\{a_i - a_{i+1}\}_i$  is decreasing. Take  $k \geq 1$  large enough such that  $\frac{a_k}{2} < r^*$  and  $\frac{a_k - a_{k+1}}{a_k} = (k+2)^{-\alpha} \leq \frac{c}{3}$ , then for  $x = a_k$  and  $r = \frac{a_k}{2}$ , we can not find such  $E_{x,r}$ . In fact, for any subset  $F$  of  $E \cap B(x, r)$  containing  $x$ , since gap sequence is decreasing, we must have

$$\text{dist}(F, E \setminus F) \leq a_k - a_{k+1} \leq \frac{ca_k}{3} < cr.$$

This is a contradiction.

(2) Secondly, we will prove that  $E$  is quasi uniformly disconnected.

For  $0 < r < 1$ , let  $\psi(r) = \frac{a_k - a_{k+1}}{2}$  if  $a_k \leq r < a_{k-1}$  for some  $k \geq 1$  with  $a_0 = 1$ . Then  $\lim_{r \rightarrow 0} \frac{\log \psi(r)}{\log r} = 1$  by (iii).

Given any  $x \in E$  and any  $0 < r < 1$ , suppose  $a_k \leq r < a_{k-1}$  for some  $k \geq 1$ . If  $0 \leq x \leq a_{k+1}$ , then we can take  $E_{x,r} = [0, a_{k+1}] \cap E$ ; if  $a_{k+1} < x \leq 1$ , then we can take  $E_{x,r} = \{x\}$ . Then

$$E \cap B(x, \psi(r)) \subset E_{x,r} \subset B(x, r) \text{ and } \text{dist}(E_{x,r}, E \setminus E_{x,r}) \geq \psi(r).$$

(3) Furthermore, we can obtain that  $\dim_H E = \dim_B E = \dim_{qA} E = 0$ .

It suffices to verify  $h_E(\delta) \leq \varepsilon$  for any fixed  $\varepsilon > 0$  and  $\delta \in (0, 1)$ .

Given  $a_k \leq r < a_{k-1}$  and  $a_l \leq R < a_{l-1}$  with  $r^{1-\delta} \leq R$ , we consider  $N_{r,R}$ . Suppose  $m_k$  is an integer such that

$$\frac{a_{m_k} - a_{m_k+1}}{2} \leq a_k < \frac{a_{m_k-1} - a_{m_k}}{2}.$$

since  $\{a_i - a_{i+1}\}_i$  is decreasing. Now we obtain

$$\lim_{k \rightarrow \infty} \frac{\log(a_{m_k}/a_k)}{\log a_{k-1}} = 0 \quad (2.1)$$

since  $\lim_{k \rightarrow \infty} \frac{\log a_{m_k}}{\log(a_{m_k} - a_{m_k+1})/2} = \lim_{k \rightarrow \infty} \frac{\log(a_{m_k} - a_{m_k+1})/2}{\log a_{k-1}} = 1$  and  $\lim_{k \rightarrow \infty} \frac{\log a_k}{\log a_{k-1}} = 1$ .

By (i)-(iii), we also have

$$\lim_{k \rightarrow \infty} \frac{\log 2m_k}{\log a_{k-1}} = \lim_{k \rightarrow \infty} \frac{\log m_k}{\log(a_{m_k} - a_{m_k+1})/2} = \lim_{m_k \rightarrow \infty} \frac{\log m_k}{\log a_{m_k}} = 0. \quad (2.2)$$

Since

$$E = ([0, a_{m_k}] \cap E) \cup ([a_{m_k-1}, 1] \cap E),$$

for  $R$  small enough, using (2.1), (2.2) and  $r^{1-\delta} \leq R$ , we obtain that

$$N_{r,R}(E) \leq N(E, a_k) \leq \frac{a_{m_k}}{2a_k} + m_k \leq \max\left\{\frac{a_{m_k}}{a_k}, 2m_k\right\} \leq (a_{k-1})^{-\delta\varepsilon} \leq r^{-\delta\varepsilon} \leq \left(\frac{R}{r}\right)^\varepsilon,$$

where  $N(E, a_k)$  is the smallest number of balls with radii  $a_k$  needed to cover  $E$ .

### 2.2. Proof of Proposition 1.

When  $X$  is fixed, we use  $N_{r,R}$  and  $N(r)$  to represent  $N_{r,R}(X)$  and  $N_{r,|X|}(X)$  respectively.

*Proof of (1) in Proposition 1.*

It is clear that  $\dim_{qA} X \leq \dim_A X$ . Now we shall verify that for any  $\delta \in (0, 1)$ ,

$$\overline{\dim}_B X \leq h_X(\delta).$$

In fact, for fixed  $\delta \in (0, 1)$ , we can assume that for any  $\alpha > h_X(\delta)$  there are  $b, c > 0$  such that for  $0 < r < r^{1-\delta} \leq R < b$ ,

$$N_{r,R} \leq c \left( \frac{R}{r} \right)^\alpha.$$

Fix some  $R < b$ . When  $r$  is small enough, using  $N(r) \leq N_{r,R} \cdot N(R)$ , we have

$$N(r) \leq N(R) \cdot c \left( \frac{R}{r} \right)^\alpha,$$

which implies

$$\frac{\log N(r)}{-\log r} \leq \frac{\log N(R)}{-\log r} + \frac{\log c}{-\log r} + \frac{\alpha \log R}{-\log r} + \alpha.$$

Letting  $r \rightarrow 0$ , we obtain that  $\overline{\dim}_B(X) \leq \alpha$  and thus  $\overline{\dim}_B(X) \leq h_X(\delta)$ .  $\square$

*Proof of (2) in Proposition 1.*

It suffices to show that  $\dim_{qA} F \leq \dim_{qA} E$ .

Suppose  $g : E \rightarrow F$  is a quasi-Lipschitz mapping. Then there exist increasing functions  $\phi, \zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with

$$\lim_{r \rightarrow 0} \frac{\log \phi(r)}{\log r} = \lim_{r \rightarrow 0} \frac{\log \zeta(r)}{\log r} = 1 \quad (2.3)$$

such that for all  $y \in F$ ,

$$B(y, R) \subset g(B(g^{-1}y, \phi(R))) \text{ and } g(B(g^{-1}y, \zeta(r))) \subset B(y, r). \quad (2.4)$$

Given  $\delta \in (0, 1/2)$  and any  $\alpha_\delta > h_E(\delta)$ , for all  $r < r^{1-2\delta} \leq R$  with  $R$  small enough, using (2.4) we obtain

$$N_{r,R}(F) \leq N_{\zeta(r), \phi(R)}(E) \leq c_\delta \left( \frac{\phi(R)}{\zeta(r)} \right)^{\alpha_\delta},$$

where  $c_\delta > 0$  is a constant and  $\zeta(r)^{1-\delta} \leq \phi(R)$  due to  $r^{1-2\delta} \leq R$  and (2.3). Since  $\frac{\log \phi(R)}{\log R} \rightarrow 1$ ,  $\frac{\log \zeta(r)}{\log r} \rightarrow 1$  and  $0 < \frac{\log R}{\log r} \leq 1 - 2\delta$ , we have

$$\frac{\log \phi(R) - \log \zeta(r)}{\log R - \log r} \rightarrow 1 \text{ as } R \rightarrow 0,$$

which implies that for any fixed  $\varepsilon > 0$ ,

$$N_{r,R}(F) \leq c_\delta \left( \frac{\phi(R)}{\zeta(r)} \right)^{\alpha_\delta} \leq c_\delta \left( \frac{R}{r} \right)^{\alpha_\delta(1+\varepsilon)}$$

whenever  $R$  is small enough. Then means  $h_F(2\delta) \leq h_E(\delta)$ . Letting  $\delta \rightarrow 0$ , we have  $\dim_{qA} F \leq \dim_{qA} E$ .  $\square$

*Proof of (3) in Proposition 1.*

We shall verify that for  $\delta \in (0, 1)$ ,

$$h_X(\delta) = \overline{\lim}_{r \rightarrow 0} \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r}. \quad (2.5)$$

For any fixed number  $\alpha > h_X(\delta)$ , there exist  $b, c > 0$  such that for all  $0 < r < r^{1-\delta} \leq R < b$ ,

$$N_{r,R} \leq c \left( \frac{R}{r} \right)^\alpha.$$

When  $r$  is small enough, if  $R \geq b$ , then  $N_{r,R} \leq N_{r, \frac{b}{2}} \cdot N_{\frac{b}{2}, R} \leq N_{r, \frac{b}{2}} \cdot N(\frac{b}{2})$ . That means for  $r^{1-\delta} \leq R < |X|$ , we obtain that

$$N_{r,R} \leq \begin{cases} c \left( \frac{R}{r} \right)^\alpha & \text{if } R < b, \\ c N(\frac{b}{2}) \left( \frac{b}{2r} \right)^\alpha & \text{if } R \geq b. \end{cases}$$

Therefore,

$$\begin{aligned} & \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r} \\ & \leq \alpha + \max \left\{ \sup_{r^{1-\delta} \leq R < b} \frac{\log c}{\log R - \log r}, \sup_{b \leq R} \frac{\log c + \log N(\frac{b}{2}) - \alpha \log 2}{\log b - \log r} \right\}, \end{aligned}$$

which implies  $\overline{\lim}_{r \rightarrow 0} \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r} \leq \alpha$  for any  $\alpha > h_X(\delta)$ . Hence

$$\overline{\lim}_{r \rightarrow 0} \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r} \leq h_X(\delta).$$

On the other hand, for any  $\alpha_0 > \overline{\lim}_{r \rightarrow 0} \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r}$ , there exists  $r_0 \in (0, 1)$  such that for  $r < r_0$ ,

$$\alpha_0 > \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r},$$

Hence for  $0 < r < r^{1-\delta} \leq R < r_0$ , we obtain that

$$\frac{\log N_{r,R}}{\log R - \log r} < \alpha_0,$$

which implies  $N_{r,R} \leq \left( \frac{R}{r} \right)^{\alpha_0}$ . Therefore  $\alpha_0 \geq h_X(\delta)$ , and thus

$$\overline{\lim}_{r \rightarrow 0} \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r} \geq h_X(\delta).$$

□

We will use the idea of [20] by Mattila and Saaranen.

*Proof of (4) in Proposition 1.*

Suppose  $\dim_{qA} X = s < 1$ . Without loss of generality, we assume that

$$|X| < 1.$$

For any  $\delta \in (0, 1)$  and  $r \in (0, 1)$ , let

$$h_X(\delta, r) = \sup_{r^{1-\delta} \leq R < |X|} \frac{\log N_{r,R}}{\log R - \log r},$$



Then by (2.5), we have  $\overline{\lim}_{r \rightarrow 0} h_X(\delta, r) = h_X(\delta)$ .

For any  $\delta \in (0, 1)$ , there exists  $r_*(\delta) \in (0, 1)$  such that for all  $r \in (0, r_*(\delta))$ ,

$$h_X(\delta, r) < h_X(\delta) + \frac{1-s}{2} \leq s + \frac{1-s}{2} = \frac{1}{2}(1+s).$$

Hence when  $r \in (0, r_*(\delta))$  and  $r^{1-\delta} \leq R < |X|$ , we always have

$$N_{r,R} \leq \left(\frac{R}{r}\right)^{\frac{1}{2}(1+s)}. \quad (2.6)$$

For any  $0 < R < |X|$ , let

$$\rho(R) = \inf_{0 < \delta < 1} \max\left\{\frac{\log r_*(\delta)}{\log R} - 1, \frac{\delta}{\delta - 1}, \frac{2 \log 6}{-(1-s) \log R}\right\}.$$

Then we have

$$\rho(R) \geq \frac{2 \log 6}{-(1-s) \log R} > 0.$$

Fix arbitrary  $\delta \in (0, 1)$ ,

$$\overline{\lim}_{R \rightarrow 0} \rho(R) \leq \overline{\lim}_{R \rightarrow 0} \max\left\{\frac{\log r_*(\delta)}{\log R} - 1, \frac{\delta}{\delta - 1}, \frac{2 \log 6}{-(1-s) \log R}\right\} \leq \frac{\delta}{\delta - 1}.$$

Letting  $\delta \rightarrow 0$ , we have

$$\lim_{R \rightarrow 0} \rho(R) = 0.$$

Fix  $x \in X$  and  $R$  small enough, let  $B_0 = \{x\}$  and

$$B_i = B(x, iR^{1+2\rho(R)}) \setminus B(x, (i-1)R^{1+2\rho(R)}) \text{ for any } i \in \mathbb{N}.$$

Write  $n_R = [R^{-2\rho(R)}]$ , where  $[z]$  denotes the integral part of  $z$ . Then  $n_R \geq [R^{\frac{4 \log 6}{(1-s) \log R}}] = [6^{\frac{4}{1-s}}] > 1$ .

We conclude that

**Claim 1.** *If  $R$  is small enough, then there exists  $1 \leq i \leq n_R$  such that*

$$B_i \cap X = \emptyset.$$

Otherwise, we assume that for all  $1 \leq i \leq n_R$ ,  $B_i \cap X$  is non-empty. We take  $y_i \in B_i \cap X$  and let

$$\Theta = \{y_i : 1 \leq i \leq n_R\}.$$

Write  $r = R^{1+2\rho(R)}$ . For any  $1 \leq i, j \leq n_R$  with  $j \geq i+3$ , the closed ball

$$\begin{aligned} B(y_j, r) &\subset X \setminus B(x, (j-2)r), \\ B(y_i, r) &\subset B(x, (i+1)r), \end{aligned}$$

which implies  $B(y_j, r) \cap B(y_i, r) = \emptyset$ , then any ball with radius  $R^{1+2\rho(R)}$  will cover at most 3 points in  $\Theta$ , i.e.,

$$\frac{n_R}{3} \leq N_{R^{1+2\rho(R)}, R}. \quad (2.7)$$

By the definition of  $\rho(R)$ , there exists  $0 < \delta < 1$  such that

$$\max\left\{\frac{\log r_*(\delta)}{\log R} - 1, \frac{\delta}{\delta - 1}, \frac{2 \log 6}{-(1-s) \log R}\right\} < 2\rho(R),$$

Therefore

$$R^{1+2\rho(R)} < r_*(\delta), \quad R^{(1+2\rho(R))} < R^{(1+2\rho(R))(1-\delta)} < R,$$

and

$$\frac{\log 6}{-(1-s)\log R} < \rho(R). \quad (2.8)$$

Let  $r = R^{1+2\rho(R)}$ , by (2.6) we have

$$N_{R^{1+2\rho(R)}, R} \leq R^{-(1+s)\rho(R)}.$$

Using (2.7) and the definition of  $n_R$ , we obtain that

$$\frac{R^{-2\rho(R)}}{6} \leq \frac{n_R}{3} \leq R^{-(1+s)\rho(R)},$$

which implies

$$\rho(R) \leq \frac{\log 6}{-(1-s)\log R}.$$

This is contradictory to (2.8). The claim is proved.

Using Claim 1, take  $\psi(R) = R^{1+2\rho(R)}$  and  $1 \leq i \leq n_R$  such that  $B_i \cap X$  is non-empty, then let

$$E_{x,R} = X \cap B(x, (i-1)\psi(R)),$$

satisfying

$$X \cap B(x, \psi(R)) \subset E_{x,R} \subset B(x, R) \text{ and } d_X(E_{x,R}, X \setminus E_{x,R}) \geq \psi(R).$$

Therefore the quasi uniform disconnectedness is obtained.  $\square$

### 3. QUASI-ASSOUAD DIMENSION OF MORAN SET

We call the closed interval  $J_{i_1 \dots i_k}$  a basic interval of rank  $k$ . For  $1 \leq p \leq q$ , let

$$s_{p,q} = \frac{\log(n_p \cdots n_q)}{-\log(c_p \cdots c_q)}. \quad (3.1)$$

*Proof of Theorem 1.*

Notice that  $\dim_{qA} E = \lim_{\delta \rightarrow 0} h_E(\delta)$ . It suffices to show

$$h_E(\delta) = \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q}, \text{ for any } \delta \in (0, 1).$$

**Step 1.** We shall verify  $h_E(\delta) \leq \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q}$ .

For any  $s > \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q}$ , we will find  $b, c > 0$  such that for all  $0 < r < r^{1-\delta} \leq R < b$ ,

$$N_{r,R} \leq c \left( \frac{R}{r} \right)^s.$$

Since  $s > \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q}$ , we can find a constant  $\sigma > 0$  small enough such that  $s(1-\sigma) > \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q}$ . Then there exists  $N \in \mathbb{N}$  such that for all  $q \geq N$  and all  $k \geq N$ ,

$$s(1-\sigma) > s_{p,q} \text{ for any } 1 \leq p \leq l_{q,\delta}, \quad (3.2)$$

and

$$\frac{\log c_k}{\log c_1 \cdots c_k} < \frac{\delta \sigma}{2}.$$

Take  $b = c_1 \cdots c_N$  and  $c = 3$ . For all  $0 < r < r^{1-\delta} \leq R < b$ , we assume that

$$c_1 \cdots c_p \leq R < c_1 \cdots c_{p-1} \text{ and } c_1 \cdots c_q \leq r < c_1 \cdots c_{q-1},$$

where  $q \geq p \geq N + 1$ . Any ball of radius  $R$  will intersect at most 3 basic interval of rank  $(p - 1)$ , i.e.,

$$N_{r,R} \leq 3n_p \cdots n_q \leq 3 \left( \frac{R}{r} \right)^{\frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q + \log c_p c_q}}.$$

Since  $(c_1 \cdots c_q)^{1-\delta} \leq r^{1-\delta} \leq R < c_1 \cdots c_{p-1}$ , we have

$$\frac{\log c_p \cdots c_q}{\log c_1 \cdots c_q} > \delta.$$

That means  $1 \leq p \leq l_{q,\delta}$ , using (3.2) we have

$$s_{p,q} = \frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q} < s(1 - \sigma).$$

On the other hand,

$$\begin{aligned} \frac{\log c_p c_q}{\log c_p \cdots c_q} &= \frac{\log c_p + \log c_q}{\log c_1 \cdots c_q} \cdot \frac{\log c_1 \cdots c_q}{\log c_p \cdots c_q} \\ &\leq \left( \frac{\log c_p}{\log c_1 \cdots c_p} + \frac{\log c_q}{\log c_1 \cdots c_q} \right) \cdot \frac{1}{\delta} \\ &< 2 \cdot \frac{\delta \sigma}{2} \cdot \frac{1}{\delta} = \sigma. \end{aligned}$$

We obtain

$$\frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q + \log c_p c_q} \leq \frac{\log n_p \cdots n_q}{-(1 - \sigma) \log c_p \cdots c_q} = \frac{s_{p,q}}{1 - \sigma} < s.$$

Therefore,

$$N_{r,R} \leq 3 \left( \frac{R}{r} \right)^s = c \left( \frac{R}{r} \right)^s.$$

**Step 2.** We shall verify  $h_E(\delta) \geq \lim_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q}$  for any  $\delta \in (0, 1)$ .

Fix  $\delta \in (0, 1)$ , we assume that  $\alpha > h_E(\delta)$ , then there exist  $b, c > 0$  such that for all  $0 < r < r^{1-\delta} \leq R < b$ ,

$$N_{r,R} \leq c \left( \frac{R}{r} \right)^\alpha.$$

Without loss of generality, we assume that there exists  $M \in \mathbb{N}$  such that

$$c_1 \cdots c_M < b \leq c_1 \cdots c_{M-1}.$$

For any fixed number  $\varepsilon \in (0, \frac{2(1-\delta)}{\delta})$ , we can take  $N \in \mathbb{N}$  large enough such that for all  $q \geq N \geq M$ ,

$$\frac{\log c_1 \cdots c_M}{\log c_1 \cdots c_q} < \frac{\varepsilon}{2 + \varepsilon} \text{ and } \frac{\log 4c}{-\log c_1 \cdots c_q} < \frac{\delta \varepsilon}{2}.$$

For any  $q \geq N$ , let  $r = c_1 \cdots c_q$ . Since

$$\frac{\log c_{M+1} \cdots c_q}{\log c_1 \cdots c_q} = 1 - \frac{\log c_1 \cdots c_M}{\log c_1 \cdots c_q} > 1 - \frac{\varepsilon}{2 + \varepsilon} > \delta,$$

then  $M + 1 \leq l_{q,\delta}$ . For any integer  $p \in [M + 1, l_{q,\delta}]$ , let  $R = c_1 \cdots c_{p-1}$ . Then  $0 < r < r^{1-\delta} \leq R < b$ , and thus

$$\frac{n_p \cdots n_q}{4} \leq N_{r,R} \leq c \left( \frac{R}{r} \right)^\alpha = c \left( \frac{1}{c_p \cdots c_q} \right)^\alpha,$$

which implies for  $M < p \leq l_{j,q}$ ,

$$\begin{aligned} \frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q} &\leq \frac{\log 4c}{-\log c_p \cdots c_q} + \alpha \\ &= \frac{\log 4c}{-\log c_1 \cdots c_q} \cdot \frac{\log c_1 \cdots c_q}{\log c_p \cdots c_q} + \alpha < \frac{\delta\varepsilon}{2} \cdot \frac{1}{\delta} + \alpha = \alpha + \frac{\varepsilon}{2}. \end{aligned}$$

When  $p \leq M$ , using  $n_k \leq c_k^{-1}$  for all  $k$ , we obtain

$$\begin{aligned} \frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q} &= \frac{\log n_p \cdots n_M}{-\log c_p \cdots c_q} + \frac{\log n_{M+1} \cdots n_q}{-\log c_p \cdots c_q} \\ &\leq \frac{\log c_p \cdots c_M}{\log c_p \cdots c_q} + (\alpha + \frac{\varepsilon}{2}) \\ &\leq \frac{\log c_1 \cdots c_M}{\log c_{M+1} \cdots c_q} + (\alpha + \frac{\varepsilon}{2}) \\ &= \frac{\log c_1 \cdots c_M}{\log c_1 \cdots c_q} \cdot \frac{\log c_1 \cdots c_q}{\log c_{M+1} \cdots c_q} + (\alpha + \frac{\varepsilon}{2}) \\ &< \frac{\varepsilon}{2 + \varepsilon} \cdot \frac{2 + \varepsilon}{2} + (\alpha + \frac{\varepsilon}{2}) = \alpha + \varepsilon. \end{aligned}$$

Therefore

$$\max_{1 \leq p \leq l_{q,\delta}} s_{p,q} \leq \alpha + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\max_{1 \leq p \leq l_{q,\delta}} s_{p,q} \leq \alpha,$$

and thus  $\overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q} \leq \alpha$ . Therefore

$$\overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq l_{q,\delta}} s_{p,q} \leq h_E(\delta)$$

follows.  $\square$

#### 4. RESULT OF GENERAL MORAN SET

Consider general Moran set  $E \in \mathcal{M}\{J, \{n_k\}_k, \{c_{k,i}\}_{k \geq 1, 1 \leq i \leq n_k}\}$  in Euclidean space  $\mathbb{R}^d$  with  $d \geq 1$ , see [29] for details of general Moran set.

For any  $p \leq q$ , let  $s_{p,q}$  be the unique positive solution of the equation  $\Delta_{p,q}(s) = 1$ , where

$$\Delta_{p,q}(s) = \prod_{i=p}^q \left( \sum_{j=1}^{n_i} (c_{i,j})^s \right). \quad (4.1)$$

In fact, if  $c_{k,1} = \cdots = c_{k,n_k} = c_k$  for all  $k$ , then  $s_{p,q} = \frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q}$  as in (3.1). Using the method in [13], we can obtain the following result and skip its proof.

**Proposition 2.** *If  $\inf_{k,i} c_{k,i} > 0$ , then for any  $E \in \mathcal{M}\{J, \{n_k\}_k, \{c_{k,i}\}_{k \geq 1, 1 \leq i \leq n_k}\}$ , we obtain that*

$$\dim_{qA} E = \lim_{\eta \rightarrow 0} \overline{\lim}_{q \rightarrow \infty} \max_{1 \leq p \leq q(1-\eta)} s_{p,q}.$$

When  $c_{k,1} = \cdots = c_{k,n_k} = c_k$  for all  $k$  and  $\inf_k c_k > 0$ , the result is the same as in Theorem 1 since  $s_{p,q} = \frac{\log n_p \cdots n_q}{-\log c_p \cdots c_q}$ .

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